

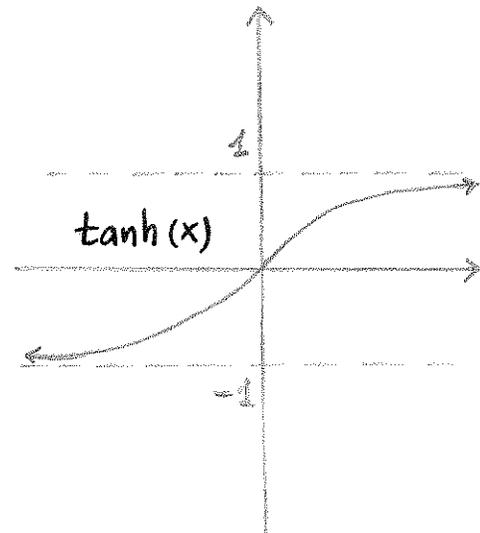
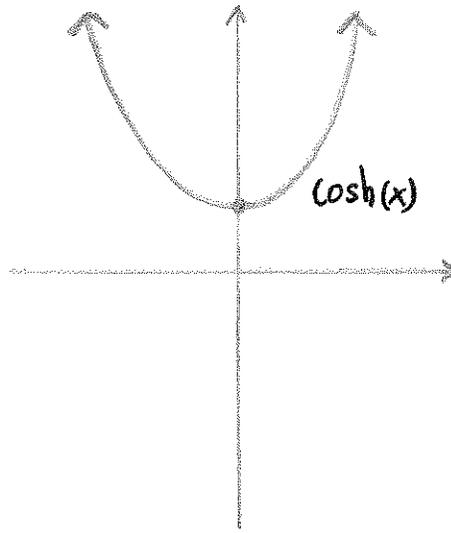
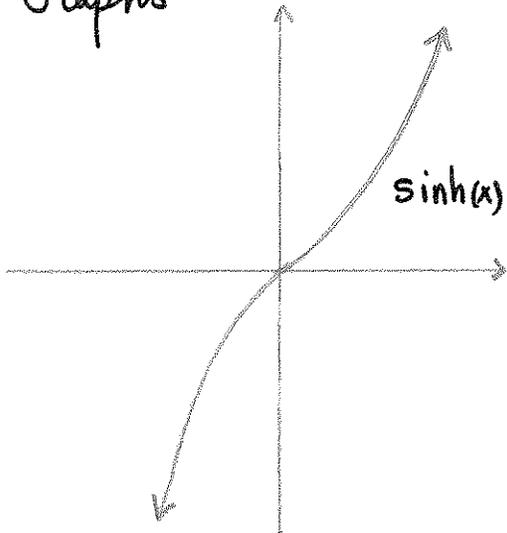
Section 6.7 Hyperbolic Functions

Definition of the hyperbolic functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}; \quad \cosh(x) = \frac{e^x + e^{-x}}{2}; \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}; \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)}; \quad \operatorname{coth}(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Graphs



Domain = Range = \mathbb{R}

Domain = \mathbb{R} , Range = $[1, \infty)$

Domain = \mathbb{R} , Range = $(-1, 1)$

Applications: modeling "hanging wires" and velocities of "ocean waves".

Hyperbolic Identities

① $\sinh(-x) = -\sinh(x)$ (\sinh is odd) $\cosh(-x) = \cosh(x)$ (\cosh is even)

② $\cosh^2(x) - \sinh^2(x) = 1$

③ $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

④ $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

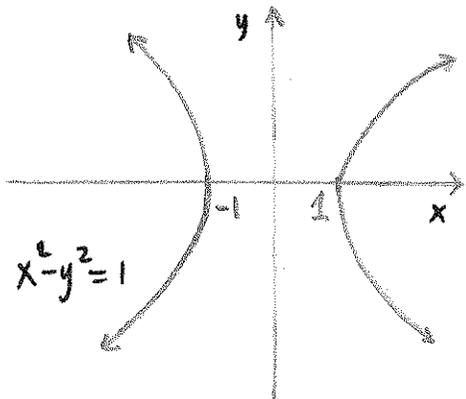
⑤ $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Proof ② $\cosh^2(x) - \sinh^2(x) = \frac{1}{4}(e^{2x} + 2e^x e^{-x} + e^{-2x}) - \frac{1}{4}(e^{2x} - 2e^x e^{-x} + e^{-2x}) = 1$

③ $\frac{\cosh^2(x)}{\cosh^2(x)} - \frac{\sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} \Rightarrow 1 - \tanh^2(x) = \operatorname{sech}^2(x)$ ■

Where does the name "Hyperbolic" come from?

First recall that $x^2 - y^2 = 1$ is the Equation of a Hyperbola



If we consider the "parametric equations"

$$\left\{ \begin{array}{l} x(t) = \cosh(t), \text{ and } y(t) = \sinh(t) \end{array} \right\},$$

Then $x^2(t) - y^2(t) = \cosh^2(t) - \sinh^2(t) = 1$ for any t .

This means that for any value of t , the point $(\cosh(t), \sinh(t))$ is on the hyperbola $x^2 - y^2 = 1$.

Derivatives: we just use the definitions of hyperbolic functions.

Ex $\frac{d}{dx} \sinh(x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$

$\frac{d}{dx} \cosh(x) = \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right) = \frac{e^x - e^{-x}}{2} = \sinh(x)$.

$F(x)$	$F'(x)$	$F(x)$	$F'(x)$
$\sinh(x)$	$\cosh(x)$	$\operatorname{csch}(x)$	$-\operatorname{csch}(x) \operatorname{coth}(x)$
$\cosh(x)$	$\sinh(x)$	$\operatorname{sech}(x)$	$-\operatorname{sech}(x) \tanh(x)$
$\tanh(x)$	$\operatorname{sech}^2(x)$	$\operatorname{coth}(x)$	$-\operatorname{csch}^2(x)$

Examples ① Simplify $10 \cosh(3 \ln(x))$

$$10 \cosh(3 \ln(x)) = 10 \cdot \frac{1}{2} (e^{3 \ln x} + e^{-3 \ln x}) = 5 (e^{\ln x^3} + e^{\ln x^{-3}}) = 5(x^3 + x^{-3}).$$

② If $\cosh(x) = 5$, what is $\tanh(x)$? Assume $x > 0$.

$$\cosh^2(x) - \sinh^2(x) = 1 \Rightarrow \sinh^2(x) = \cosh^2(x) - 1 = 25 - 1 = 24.$$

Thus, $\sinh(x) = \pm \sqrt{24}$. Since $x > 0$, $\sinh(x) > 0$ and then $\sinh(x) = +\sqrt{24}$.

$$\text{Finally, } \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\sqrt{24}}{5}.$$

③ Differentiate $\ln(\tanh(x^2+1))$.

$$\begin{aligned} \frac{d}{dx} \ln(\tanh(x^2+1)) &= \frac{1}{\tanh(x^2+1)} \cdot \frac{d}{dx} \tanh(x^2+1) \quad \text{by Chain Rule} \\ &= \coth(x^2+1) \cdot \operatorname{sech}^2(x^2+1) \cdot \frac{d}{dx}(x^2+1) \quad \text{by chain rule} \\ &= \coth(x^2+1) \cdot \operatorname{sech}^2(x^2+1) \cdot 2x \end{aligned}$$

④ Evaluate $\int \tanh(x) dx$.

$$\begin{aligned} \int \tanh(x) dx &= \int \frac{\sinh(x)}{\cosh(x)} dx. \quad \text{Let } u = \cosh(x), \quad du = \sinh(x) dx \\ &= \int \frac{du}{u} = \ln |u| + C = \ln |\cosh(x)| + C. \end{aligned}$$